

Geometry of \mathcal{PR} -semi-invariant warped product submanifolds in paracosymplectic manifold

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Abstract. The purpose of this paper is to study \mathcal{PR} -semi-invariant warped product submanifolds of a paracosymplectic manifold \widetilde{M} . We prove that the distributions associated with the definition of \mathcal{PR} -semi-invariant warped product submanifold M are always integrable. A necessary and sufficient condition for an isometrically immersed \mathcal{PR} -semi-invariant submanifold of \widetilde{M} to be a \mathcal{PR} -semi-invariant warped product submanifold is obtained in terms of the shape operator.

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1. Introduction

The concept of warped product (or, more generally warped bundle) is one of the most effective generalizations of pseudo-Riemannian products [17]. The premise of warped product has perceived several important contributions in complex and contact Riemannian (or pseudo-Riemannian) geometries, and has been successfully applied in Hamiltonian spaces, general relativity and black holes (c.f., [1, 3, 7, 13]).

The study of warped product was initiated by Bishop-Neill [2]. However, the consideration has attained momentum when Chen introduced the notion of CR-warped product in Kaehlerian manifold \widetilde{N} and proved the non-existence

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of proper warped product CR-submanifolds in the form $N_T \times_f N_\perp$ such that N_T is a holomorphic submanifold and N_\perp is a totally real submanifold of \tilde{N} [4]. Subsequently, Hasegawa-Mihai [11] and Munteanu [16] continued the study for Sasakian manifold that can be viewed as an odd-dimensional analogue of \tilde{N} . Further, several geometers have studied the existence and non-existence of warped product submanifolds in almost contact and Lorentzian manifolds (c.f., [6, 14, 15, 18, 20]). Recently in [5], Chen-Munteanu brought our attention to the geometry of \mathcal{PR} -warped products in para-Kähler manifolds and obtained some basic results on such submanifolds.

This paper is organized as follows. In Sect. 2, the basic information about almost paracontact metric manifolds, paracosymplectic manifolds and submanifolds is given. In Sect. 3, we proved the non-existence of a proper warped product submanifold of a paracosymplectic manifold \tilde{M} in the form $B \times_f F$ such that the characteristic vector field ξ is tangent to F , where f is a warping function. In Sect. 4, we study \mathcal{PR} -semi-invariant warped product submanifolds of \tilde{M} and found the distributions concerned with the definition of \mathcal{PR} -semi-invariant submanifold M are integrable. Further, we obtained a necessary and sufficient condition for an isometrically immersed submanifold M of \tilde{M} to be a \mathcal{PR} -semi-invariant warped product submanifold. Finally, we gave an example of a \mathcal{PR} -semi-invariant submanifold $F \times_f B$ of a paracosymplectic manifold in Sect. 5.

2. Preliminaries

2.1. Almost paracontact metric manifolds

A $(2n + 1)$ -dimensional C^∞ manifold \tilde{M} has an almost paracontact structure (ϕ, ξ, η) , if it admits a tensor field ϕ of type $(1, 1)$, a vector field ξ , and a 1-form η on \tilde{M} , satisfying conditions:

$$\phi^2 = Id - \eta \otimes \xi, \quad \eta(\xi) = 1 \quad (2.1)$$

where Id is the identity transformation and the tensor field ϕ induces an almost paracomplex structure on the distribution $D = \ker(\eta)$, that is the eigen distributions D^\pm corresponding to the eigenvalues ± 1 , have equal dimensions $\dim D^+ = \dim D^- = n$. From the equation (2.1), it can be easily deduced that

$$\phi\xi = 0, \quad \eta \circ \phi = 0 \quad \text{and} \quad \text{rank}(\phi) = 2n. \quad (2.2)$$

The manifold \tilde{M} is said to be an almost paracontact manifold if it is endowed with an almost paracontact structure (c.f., [19, 22]). If an almost paracontact

manifold \widetilde{M} admits a pseudo-Riemannian metric g satisfying:

$$g(X, Y) = -g(\phi X, \phi Y) + \eta(X) \eta(Y), \quad (2.3)$$

where signature of g is necessarily $(n+1, n)$ for any vector fields X and Y ; then the quadruple (ϕ, ξ, η, g) is called an almost paracontact metric structure and the manifold \widetilde{M} equipped with paracontact metric structure is called an almost paracontact metric manifold. With respect to g , η is metrically dual to ξ , that is

$$g(X, \xi) = \eta(X). \quad (2.4)$$

With the consequences of Eqs. (2.1) and (2.2), Eq. (2.3) implies that

$$g(\phi X, Y) = -g(X, \phi Y), \quad (2.5)$$

for any $X, Y \in \Gamma(T\widetilde{M})$; $\Gamma(T\widetilde{M})$ denote the sections of tangent bundle $T\widetilde{M}$ of \widetilde{M} , that is, the space of vector fields on \widetilde{M} . The fundamental 2-form Φ on \widetilde{M} is given by

$$g(X, \phi Y) = \Phi(X, Y). \quad (2.6)$$

An almost paracontact metric structure (ϕ, ξ, η, g) is para-Sasakian if and only if

$$(\widetilde{\nabla}_X \phi)Y = -g(X, Y)\xi + \eta(Y)X. \quad (2.7)$$

From (2.2), (2.3) and (2.7), it can be easily deduced for a para-Sasakian manifold that

$$\widetilde{\nabla}_X \xi = -\phi X, \quad \widetilde{\nabla}_\xi \xi = 0. \quad (2.8)$$

In particular, a para-Sasakian manifold is K -paracontact [22].

Definition 2.1. An almost paracontact metric manifold $\widetilde{M}(\phi, \xi, \eta, g)$ is said to be

- paracosymplectic if the forms η and Φ are parallel with respect to the Levi-Civita connection $\widetilde{\nabla}$ on \widetilde{M} , i.e.,

$$\widetilde{\nabla}\eta = 0 \quad \text{and} \quad \widetilde{\nabla}\Phi = 0. \quad (2.9)$$

- an almost paracosymplectic if the forms η and Φ are closed, i.e., $d\eta = 0$ and $d\Phi = 0$ (see [8, 10]).

Now, we give an example of a paracosymplectic manifold:

Example 2.2. We consider the 5-dimensional manifold $\widetilde{M} = \mathbb{R}^4 \times \mathbb{R}_+ \subset \mathbb{R}^5$ with the standard Cartesian coordinates (x_1, x_2, y_1, y_2, t) . Define the structure (ϕ, ξ, η) on \widetilde{M} by

$$\phi e_1 = e_3, \quad \phi e_2 = e_4, \quad \phi e_3 = e_1, \quad \phi e_4 = e_2, \quad \phi e_5 = 0, \quad \xi = e_5, \quad \eta = dt, \quad (2.10)$$

where $e_1 = \frac{\partial}{\partial x_1}$, $e_2 = \frac{\partial}{\partial x_2}$, $e_3 = \frac{\partial}{\partial y_1}$, $e_4 = \frac{\partial}{\partial y_2}$ and $e_5 = \frac{\partial}{\partial t}$. Consider g to be the pseudo-Riemannian metric defined by

$$[g(e_i, e_j)] = \begin{bmatrix} x^2 & 0 & 0 & 0 & 0 \\ 0 & y^2 & 0 & 0 & 0 \\ 0 & 0 & -x^2 & 0 & 0 \\ 0 & 0 & 0 & -y^2 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}. \quad (2.11)$$

Then by straightforward calculations, one verifies that the structure (ϕ, ξ, η, g) is an almost paracontact metric structure. For the Levi-Civita connection $\tilde{\nabla}$ with respect to pseudo-Riemannian metric g , we obtain

$$\begin{aligned} \tilde{\nabla}_{e_1} e_1 &= \frac{1}{x} e_1, & \tilde{\nabla}_{e_1} e_2 &= 0, & \tilde{\nabla}_{e_1} e_3 &= \frac{1}{x} e_3, & \tilde{\nabla}_{e_1} e_4 &= 0, & \tilde{\nabla}_{e_1} e_5 &= 0, \\ \tilde{\nabla}_{e_2} e_1 &= 0, & \tilde{\nabla}_{e_2} e_2 &= \frac{1}{y} e_2, & \tilde{\nabla}_{e_2} e_3 &= 0, & \tilde{\nabla}_{e_2} e_4 &= \frac{1}{y} e_4, & \tilde{\nabla}_{e_2} e_5 &= 0, \\ \tilde{\nabla}_{e_3} e_1 &= \frac{1}{x} e_3, & \tilde{\nabla}_{e_3} e_2 &= 0, & \tilde{\nabla}_{e_3} e_3 &= \frac{1}{x} e_1, & \tilde{\nabla}_{e_3} e_4 &= 0, & \tilde{\nabla}_{e_3} e_5 &= 0, \\ \tilde{\nabla}_{e_4} e_1 &= 0, & \tilde{\nabla}_{e_4} e_2 &= \frac{1}{y} e_4, & \tilde{\nabla}_{e_4} e_3 &= 0, & \tilde{\nabla}_{e_4} e_4 &= \frac{1}{y} e_2, & \tilde{\nabla}_{e_4} e_5 &= 0, \\ \tilde{\nabla}_{e_5} e_1 &= 0, & \tilde{\nabla}_{e_5} e_2 &= 0, & \tilde{\nabla}_{e_5} e_3 &= 0, & \tilde{\nabla}_{e_5} e_4 &= 0, & \tilde{\nabla}_{e_5} e_5 &= 0. \end{aligned}$$

From the above computations, it can be easily seen that $\tilde{M}(\phi, \xi, \eta, g)$ is a parasymplectic manifold.

2.2. Geometry of submanifolds

Let M be a submanifold immersed in a $(2n+1)$ -dimensional almost paracontact manifold \tilde{M} ; we denote by the same symbol g the induced metric on M . Let $\Gamma(TM^\perp)$ denote the set of vector fields normal to M and $\Gamma(TM)$ the sections of tangent bundle TM of M then Gauss and Weingarten formulas are given by, respectively,

$$\tilde{\nabla}_X Y = \nabla_X Y + h(X, Y), \quad (2.12)$$

$$\tilde{\nabla}_X \zeta = -A_\zeta X + \nabla_X^\perp \zeta \quad (2.13)$$

for any $X, Y \in \Gamma(TM)$ and $\zeta \in \Gamma(TM^\perp)$, where ∇ is the induced connection, ∇^\perp is the normal connection on the normal bundle TM^\perp , h is the second fundamental form, and the shape operator A_ζ associated with the normal section ζ is given by

$$g(A_\zeta X, Y) = g(h(X, Y), \zeta). \quad (2.14)$$

The mean curvature vector H of M is given by $H = \frac{1}{n} \text{trace}(h)$. A pseudo-Riemannian submanifold M is said to be [5]

- *totally geodesic* if its second fundamental form vanishes identically.
- *umbilical* in the direction of a normal vector field ζ on M , if $A_\zeta = \lambda Id$, for certain function λ on M ; here ζ is called a umbilical section.

- *totally umbilical* if M is umbilical with respect to every (local) normal vector field.
- *minimal* if the mean curvature vector H vanishes identically.
- *quasi-minimal* if H is a light-like vector field.

Consider that M is an isometrically immersed submanifold of an almost paracontact metric manifold \widetilde{M} . For any $X \in \Gamma(TM)$ and $N \in \Gamma(TM^\perp)$, if we write

$$\phi X = tX + nX, \quad (2.15)$$

$$\phi N = t'N + n'N, \quad (2.16)$$

where tX (resp., nX) is tangential (resp., normal) part of ϕX and $t'N$ (resp., $n'N$) is tangential (resp., normal) part of ϕN . Then the submanifold M is said to be *invariant* if n is identically zero and *anti-invariant* if t is identically zero. From Eqs. (2.5) and (2.15), we obtain that

$$g(X, tY) = -g(tX, Y). \quad (2.17)$$

Let M be an immersed submanifold of a paracosymplectic manifold \widetilde{M} then for any $X, Y \in \Gamma(TM)$ we obtain by use of Eqs. (2.9), (2.12), (2.13) and (2.14) that

$$(\nabla_X t)Y = A_{nY}X + t'h(X, Y), \quad (2.18)$$

$$(\nabla_X n)Y = n'h(X, Y) - h(X, tY), \quad (2.19)$$

where the covariant derivatives of the tensor fields t and n are, respectively, defined by

$$(\nabla_X t)Y = \nabla_X tY - t\nabla_X Y, \quad (2.20)$$

$$(\nabla_X n)Y = \nabla_X^\perp nY - n\nabla_X Y. \quad (2.21)$$

The canonical structure t and n on a submanifold M are said to be *parallel* if $\nabla t = 0$ and $\nabla n = 0$, respectively. From Eqs. (2.9) and (2.12), we can easily prove the following lemma for later use:

Lemma 2.3. *Let M be an immersed submanifold of a paracosymplectic manifold $\widetilde{M}(\phi, \xi, \eta, g)$ such that ξ is tangent to M . Then for any $X \in \Gamma(TM)$, we have*

$$\nabla_X \xi = 0, \quad (2.22)$$

$$h(X, \xi) = 0. \quad (2.23)$$

3. Warped product submanifolds

Let (B, g_B) and (F, g_F) be two pseudo-Riemannian manifolds and f be a positive smooth function on B . Consider the product manifold $B \times F$ with canonical

projections

$$\pi : B \times F \rightarrow B \quad \text{and} \quad \sigma : B \times F \rightarrow F. \quad (3.1)$$

Then the manifold $M = B \times_f F$ is said to be *warped product* if it is equipped with the following warped metric

$$g(X, Y) = g_B(\pi_*(X), \pi_*(Y)) + (f \circ \pi)^2 g_F(\sigma_*(X), \sigma_*(Y)) \quad (3.2)$$

for all $X, Y \in \Gamma(TM)$ and $'\ast'$ stands for derivation map, or equivalently,

$$g = g_B + f^2 g_F. \quad (3.3)$$

The function f is called *the warping function* and a warped product manifold M is said to be *trivial* if f is constant [2].

Now, we recall the following proposition for the warped product manifolds [2]:

Proposition 3.1. *For $X, Y \in \Gamma(TB)$ and $U, V \in \Gamma(TF)$, we obtain on warped product manifold $M = B \times_f F$ that*

- (1) $\nabla_X Y \in \Gamma(TB)$,
- (2) $\nabla_X U = \nabla_U X = X(\ln f)U$,
- (3) $\nabla_U V = \nabla'_U V - g(U, V) \text{grad}(\ln f)$,

where ∇ and ∇' denotes the Levi-Civita connections on M and F respectively.

For a warped product $M = B \times_f F$; B is totally geodesic and F is totally umbilical in M [2].

In [9], Ehrlich introduced a notion of doubly warped product to generalize the warped product. Let us consider the product manifold $B \times F$ with canonical projections given by (3.1). Then a doubly warped product of pseudo-Riemannian manifolds of (B, g_B) and (F, g_F) with smooth warping functions $f_1 : B \rightarrow (0, \infty)$ and $f_2 : F \rightarrow (0, \infty)$ is a manifold ${}_{f_2}B \times_{f_1} F$ endowed with the following doubly warped metric

$$g(X, Y) = (f_2 \circ \sigma)^2 g_B(\pi_*(X), \pi_*(Y)) + (f_1 \circ \pi)^2 g_F(\sigma_*(X), \sigma_*(Y)) \quad (3.4)$$

for all $X, Y \in \Gamma(TM)$, or equivalently,

$$g = f_2^2 g_B + f_1^2 g_F. \quad (3.5)$$

If either $f_1 = 1$ or $f_2 = 1$, but not both, then ${}_{f_2}B \times_{f_1} F$ becomes a warped product. If $f_1 = f_2 = 1$, then we have a product manifold. If neither f_1 nor f_2 is constant, then we obtain a proper (non trivial) doubly warped product manifold (see also [18, 21]). In this case formula (2) of proposition 3.1 is generalized as

$$\nabla_X V = (X \ln f_1) V + (V \ln f_2) X \quad (3.6)$$

for each $X \in \Gamma(TB)$ and $V \in \Gamma(TF)$ [16]. For the proper doubly warped product manifold $M = {}_{f_2}B \times_{f_1} F$, we have from [21] that the:

- (i) leaves $B \times \{q\}$ and the fibers $\{p\} \times F$ of M are totally umbilical and
- (ii) leaf $B \times \{q\}$ (resp., fiber $\{p\} \times F$) is totally geodesic if $\text{grad}_F(f_2)|_q = 0$ (resp., $\text{grad}_B(f_1)|_p = 0$).

Presently we will prove the following theorem:

Theorem 3.2. *There do not exist a proper warped product submanifold $M = B \times_f F$ of a paracosymplectic manifold $\widetilde{M}(\phi, \xi, \eta, g)$ such that ξ have both TB and TF components.*

Proof. For $\xi \in \Gamma(TM)$ we can write $\xi = \xi_1 + \xi_2$, where $\xi_1 \in \Gamma(TB)$, $\xi_2 \in \Gamma(TF)$. Therefore, by the consequences of lemma 2.3 and proposition 3.1, we obtain $X(\ln f)\xi_2 = 0$, $\forall X \in \Gamma(TB)$ and $g(Z, \xi_2)\text{grad}(\ln f) = 0$, $\forall Z \in \Gamma(TF)$, both of which implies that $\xi_2 = 0$ since f is not constant. This completes the proof of the theorem. \square

Let $M = {}_{f_2}B \times_{f_1}F$ be a doubly warped product submanifold of a paracosymplectic manifold $\widetilde{M}(\phi, \xi, \eta, g)$ such that $\xi \in \Gamma(TM)$. If we consider $\xi \in \Gamma(TF)$. Then from Eqs. (3.6) and (2.22), we obtain that $\nabla_X \xi = X(\ln f_1)\xi + \xi(\ln f_2)X = 0$. This reduced to the equation $X(\ln f_1) = 0$, $\forall X \in \Gamma(TB)$ which yields f_1 is constant. Again if we take $\xi \in \Gamma(TB)$ then $Z(\ln f_2) = 0$, $\forall Z \in \Gamma(TF)$ this implies that f_2 is constant. Therefore, we can state the following proposition:

Proposition 3.3. *Let $M = {}_{f_2}B \times_{f_1}F$ be a doubly warped product submanifold of a paracosymplectic manifold $\widetilde{M}(\phi, \xi, \eta, g)$. Then f_1 (resp., f_2) is constant if $\xi \in \Gamma(TF)$ (resp., $\xi \in \Gamma(TB)$).*

As an immediate consequence of the proposition 3.3, we have

Corollary 3.4. *Let $\widetilde{M}(\phi, \xi, \eta, g)$ be a paracosymplectic manifold. Then there do not exist a proper warped product submanifold $M = B \times_f F$ of \widetilde{M} for $\xi \in \Gamma(TF)$.*

Now we prove the following important lemma for later use:

Lemma 3.5. *Let $M = B \times_f F$ be a proper warped product submanifold of a paracosymplectic manifold $\widetilde{M}(\phi, \xi, \eta, g)$ such that $\xi \in \Gamma(TB)$. Then we have*

$$\xi(\ln f) = 0, \quad (3.7)$$

$$A_{nZ}X = -t'h(X, Z), \quad (3.8)$$

$$g(h(X, W), nZ) = g(h(X, Z), nW) = -tX(\ln f)g(Z, W), \quad (3.9)$$

for any $X, Y \in \Gamma(TB)$ and $Z, W \in \Gamma(TF)$.

Proof. Equation (3.7) directly follows from Eq. (2.22) and proposition 3.1. Again by use of proposition 3.1 and Eq.(2.9), we obtain that

$$\widetilde{\nabla}_X \phi Z - \phi(\widetilde{\nabla}_X Z) = 0. \quad (3.10)$$

On employing Eqs. (2.12), (2.15) and (2.16) in Eq.(3.10), we get

$$h(X, tZ) - A_{nZ}X + \nabla_X^\perp nZ = t'h(X, Z) + n'h(X, Z). \quad (3.11)$$

By comparing the tangential part of (3.11), we have Eq. (3.8). In view of Eqs. (2.5), (2.9), (2.13), (3.8) and proposition 3.1, we achieve Eq. (3.9). This completes the proof of the lemma. \square

4. \mathcal{PR} -semi-invariant warped product

In [5], Chen-Munteanu defined \mathcal{PR} -warped products in para-Kähler manifolds. Motivated to the study of Chen-Munteanu, we define \mathcal{PR} -semi-invariant warped product submanifolds of an almost paracontact manifold.

Definition 4.1. Let M is an isometrically immersed pseudo-Riemannian submanifold of an almost paracontact manifold $\widetilde{M}(\phi, \xi, \eta, g)$. Then M is said to be a \mathcal{PR} -semi-invariant submanifold if it is furnished with the pair of orthogonal distribution $(\mathfrak{D}, \mathfrak{D}^\perp)$ satisfying the conditions:

- (i) $TM = \mathfrak{D} \oplus \mathfrak{D}^\perp \oplus \langle \xi \rangle$,
- (ii) the distribution \mathfrak{D} is invariant under ϕ , i.e., $\phi(\mathfrak{D}) = \mathfrak{D}$ and
- (iii) the distribution \mathfrak{D}^\perp is anti-invariant under ϕ , i.e., $\phi(\mathfrak{D}^\perp) \subset TM^\perp$.

A \mathcal{PR} -semi-invariant submanifold is called a \mathcal{PR} -semi-invariant warped product if it is a warped product of the form: $B \times_f F$ or $F \times_f B$, where B is an invariant submanifold, F is an anti-invariant submanifold of an almost paracontact manifold $\widetilde{M}(\phi, \xi, \eta, g)$ and f is a non-constant positive smooth function on the first factor. If the warping function f is constant then a \mathcal{PR} -semi-invariant warped product submanifold is said to be a \mathcal{PR} -semi-invariant product.

In this section we shall examine \mathcal{PR} -semi-invariant warped product submanifolds of a paracosymplectic manifold \widetilde{M} .

Proposition 4.2. *There do not exist a \mathcal{PR} -semi-invariant warped product submanifold $M = B \times_f F$ of a paracosymplectic manifold $\widetilde{M}(\phi, \xi, \eta, g)$ such that the characteristic vector field ξ is tangent to F .*

Proof. By the virtue of proposition (3.1) and Eq. (2.22), we obtain that $\nabla_X \xi = \nabla_\xi X = X(\ln f)\xi = 0$, $\forall X \in \Gamma(TB)$, which implies that f is constant on B . This completes the proof. \square

Theorem 4.3. *There do not exist a \mathcal{PR} -semi-invariant warped product submanifold $M = B \times_f F$ of a paracosymplectic manifold $\widetilde{M}(\phi, \xi, \eta, g)$ such that the characteristic vector field ξ is normal to M .*

Proof. Let $M = B \times_f F$ be a \mathcal{PR} -semi-invariant warped product in \widetilde{M} with $\xi \in \Gamma(TM^\perp)$. Then for any $X \in \Gamma(TB)$ and $Z \in \Gamma(TF)$ we obtain from proposition 3.1 that $\nabla_X Z = \nabla_Z X = X(\ln f)Z$, by taking the inner product with Z and using Eqs. (2.3), (2.5), (2.9) and Gauss formula (2.12), we get

$$X(\ln f)\|Z\|^2 = -g(h(Z, \phi X), \phi Z).$$

Interchanging Z by ϕZ , we have $X(\ln f)\|Z\|^2 = 0$. This implies that f is constant on B since Z is non-null vector field in F . This completes the proof of the theorem. \square

Proposition 4.4. *There do not exist a \mathcal{PR} -semi-invariant warped product submanifold $M = F \times_f B$ of a paracosymplectic manifold $\widetilde{M}(\phi, \xi, \eta, g)$ such that the characteristic vector field ξ is tangent to B .*

Proof. When $\xi \in \Gamma(TB)$, then by corollary 3.4 we have simply a \mathcal{PR} -semi-invariant warped product manifold. This completes the proof. \square

Now, we give the following important result:

Theorem 4.5. *Let $M = F \times_f B$ be a \mathcal{PR} -semi-invariant warped product submanifold of a paracosymplectic manifold $\widetilde{M}(\phi, \xi, \eta, g)$ such that the characteristic vector field ξ is tangent to F . Then the invariant distribution \mathfrak{D} and the anti-invariant distribution \mathfrak{D}^\perp are always integrable.*

Before going to the proof of this theorem, we first prove the following lemma:

Lemma 4.6. *For a \mathcal{PR} -semi-invariant warped product submanifold $M = F \times_f B$ of a paracosymplectic manifold $\widetilde{M}(\phi, \xi, \eta, g)$ with $\xi \in \Gamma(TF)$, we obtain for all $U, V, Z \in \Gamma(\mathfrak{D})$ and $X, Y \in \Gamma(\mathfrak{D}^\perp)$ that*

$$A_{nX}U = -X(\ln f)tU, \quad (4.1)$$

$$A_{nY}X = A_{nX}Y = t'h(X, Y) = 0, \quad (4.2)$$

$$h(U, tV) = -g(U, V)n(grad(\ln f)) + n'h(U, V) = h(V, tU). \quad (4.3)$$

Proof. It readily follows from Eqs. (2.9), (2.12), (2.15) and (2.16) that $h(X, tU) = t'h(X, U) + n'h(X, U)$. This equation yields by comparing the tangential parts that $t'h(X, U) = 0$ and by making use of equation (2.18), we have the formula (4.1). Since, the distribution \mathfrak{D}^\perp is totally geodesic in M and anti-invariant then from Eqs. (2.12) and (2.13), we get

$$-A_{nY}X + \nabla_X^\perp nY = n(\nabla_X Y) + t'h(X, Y) + n'h(X, Y). \quad (4.4)$$

By equating the tangential components of Eq. (4.4) and then interchanging X to Y , we obtain that

$$A_{nY}X = A_{nX}Y = -t'h(X, Y). \quad (4.5)$$

Employing Eqs. (2.5), (2.9)-(2.14), proposition 3.1 and using the fact that A is self-adjoint, we attain

$$g(A_{nX}Y, Z) = -g(A_{nY}X, Z). \quad (4.6)$$

In light of Eqs.(4.5) and (4.6), we obtain formula (4.2). On the other side we obtain

$$h(U, tV) + \nabla_U tV = t(\nabla'_U V) - g(U, V)n(\text{grad}(\ln f)) + t'h(U, V) + n'h(U, V). \quad (4.7)$$

By equating the normal components of Eq. (4.7), we get formula (4.3). This completes the proof. \square

Proof of Theorem 4.5. Let $U, V \in \Gamma(\mathfrak{D})$ then by the virtue of Eqs.(2.19), (4.3) and using the fact that h is symmetric, we have

$$\begin{aligned} n([V, U]) &= n(\nabla_V U) - n(\nabla_U V) = \nabla_V^\perp nU - (\nabla_V n)U - \nabla_U^\perp nV + (\nabla_U n)V \\ &= (\nabla_U n)V - (\nabla_V n)U \\ &= n'h(U, V) - h(U, tV) - n'h(U, V) + h(V, tU) = 0, \end{aligned}$$

this implies that $[V, U] \in \Gamma(\mathfrak{D})$ for any $U, V \in \mathfrak{D}$. Similarly, by using Eqs. (2.18) and (4.5) we find that $t([X, Y]) = 0$ which implies $[X, Y] \in \Gamma(\mathfrak{D}^\perp)$ for any $X, Y \in \Gamma(\mathfrak{D}^\perp)$. Thus the distributions \mathfrak{D} and \mathfrak{D}^\perp are integrable. This complete the proof of the theorem. \square

Theorem 4.7. *Let $M \rightarrow \widetilde{M}$ be an isometric immersion of a pseudo-Riemannian manifold M into a paracosymplectic manifold $\widetilde{M}(\phi, \xi, \eta, g)$ such that the characteristic vector field ξ is tangent to M . Then a necessary and sufficient condition for M to be a \mathcal{PR} -semi-invariant submanifold is that $n \circ t = 0$.*

Proof. If we denote the orthogonal projections on the invariant distribution \mathfrak{D} and the anti-invariant distribution \mathfrak{D}^\perp by P_1 and P_2 respectively. Then we have

$$P_1 + P_2 = Id, (P_1)^2 = P_1, (P_2)^2 = P_2, P_1 P_2 = P_2 P_1 = 0. \quad (4.8)$$

Since $\xi \in \Gamma(TM)$, then for any $X \in \Gamma(TM)$, $Z \in \Gamma(TM^\perp)$ we obtain

$$X - \eta(X)\xi = t^2 X + t'nX, \quad (4.9)$$

$$ntX + n'nX = 0, \quad (4.10)$$

$$tt'Z + t'n'Z = 0, \quad (4.11)$$

$$nt'Z + n'^2 Z = Z. \quad (4.12)$$

From Eqs. (2.1) and (2.15), we can write

$$\begin{aligned} X - \eta(X)\xi &= P_1X + P_2X, \\ \phi X &= \phi(P_1X) + \phi(P_2X), \\ tX + nX &= tP_1X + nP_1X + tP_2X + nP_2X \end{aligned}$$

for any $X \in \Gamma(TM)$. By comparing the tangential and the normal parts of last equation, we find

$$tX = tP_1X + tP_2X, \quad nX = nP_1X + nP_2X. \quad (4.13)$$

For the invariant distribution \mathfrak{D} and the anti-invariant distribution \mathfrak{D}^\perp , we obtain that $nP_1 = 0$ and $tP_2 = 0$. Thus from Eq. (4.13), we have

$$t = tP_1, \quad n = nP_2$$

which gives

$$ntX = nP_2tX = nP_2tP_1X = nt(P_1P_2)X = 0, \quad \forall X \in \Gamma(TM).$$

Conversely, suppose that M be submanifold of a paracosymplectic manifold \widetilde{M} such that $\xi \in \Gamma(TM)$ satisfying $nt = 0$. Then from Eq. (4.10), we have

$$n'n = 0. \quad (4.14)$$

Employing Eqs. (2.5), (4.10) and (4.14), we obtain that $g(X, tt'Z) = 0$ for any $X \in \Gamma(TM)$ and $Z \in \Gamma(TM^\perp)$ which implies that $tt' = 0$. Therefore from Eq. (4.11), we also have $t'n' = 0$. Further, from Eqs. (4.9) and (4.12), we get

$$t^3 = t, \quad n'^3 = n'. \quad (4.15)$$

By substituting

$$P_1 = t^2 \text{ and } P_2 = Id - t^2, \quad (4.16)$$

we achieve Eq. (4.8), this implies P_1 and P_2 are orthogonal complementary projections defining distributions \mathfrak{D} and \mathfrak{D}^\perp . From Eqs. (4.15) and (4.16), we determine that $t = tP_1$, $tP_2 = 0$, $n = nP_2$, $nP_1 = 0$ and $P_2tP_1 = 0$. These implies that the distribution \mathfrak{D} is invariant and the distribution \mathfrak{D}^\perp is anti-invariant, and hence completes the proof of the theorem. \square

Theorem 4.8. *Let M be a \mathcal{PR} -semi-invariant submanifold of a paracosymplectic manifold $\widetilde{M}(\phi, \xi, \eta, g)$. Then M is a \mathcal{PR} -semi-invariant warped product $F \times_f B$ iff the shape operator of M satisfies*

$$A_{\phi X}U = -X(\mu)\phi U, \quad X \in \Gamma(\mathfrak{D}^\perp), U \in \Gamma(\mathfrak{D}) \quad (4.17)$$

for some function μ on M such that $W(\mu) = 0$, $W \in \Gamma(\mathfrak{D})$.

Proof. Let $M = F \times_f B$ be a \mathcal{PR} -semi-invariant warped product submanifold of a paracosymplectic manifold \widetilde{M} then from Eq.(4.1), we obtain that $A_{\phi X}U = -X(\ln f)\phi U$ for any $X \in \Gamma(\mathfrak{D}^\perp)$ and $U \in \Gamma(\mathfrak{D})$. Since f is a function on the first factor F , putting $\mu = \ln f$ implies that $W(\mu) = 0$ for all $W \in \Gamma(\mathfrak{D})$. Conversely, assume that M satisfies (4.17) for some function μ on M with $W(\mu) = 0$, for all $W \in \Gamma(\mathfrak{D})$. By the virtue of Eqs. (2.5), (2.9)-(2.13) and (4.2), we have

$$g(\nabla_X Y, \phi V) = g(\widetilde{\nabla}_X Y, \phi V) = -g(\widetilde{\nabla}_X \phi Y, V) = -g(A_{\phi Y} X, V) = 0,$$

for any $X, Y \in \Gamma(\mathfrak{D}^\perp)$ and $V \in \Gamma(\mathfrak{D})$. Therefore the distribution \mathfrak{D}^\perp is totally geodesic. On the other hand from Eqs.(2.3) and (4.1), we get

$$\begin{aligned} g(\nabla_U V, X) &= g(\widetilde{\nabla}_U V, X) = -g(V, \widetilde{\nabla}_U X) = g(\phi V, \widetilde{\nabla}_U \phi X) \\ &= -g(\phi V, A_{nX} U) = X(\mu)g(\phi V, \phi U) = -X(\mu)g(U, V), \end{aligned}$$

for any $U, V \in \Gamma(\mathfrak{D})$, where $\mu = \ln f$. Thus, the integrable manifold of \mathfrak{D} is totally umbilical submanifold in M and its mean curvature is non-zero and parallel by using the facts that the distribution \mathfrak{D} of M is always integrable and $W(\mu) = 0$ for all $W \in \Gamma(TB)$, and hence completes the proof of the theorem. \square

Now, we prove the following result:

Theorem 4.9. *Let $M \rightarrow \widetilde{M}$ be an isometric immersion of a pseudo-Riemannian manifold M into a paracosymplectic manifold $\widetilde{M}(\phi, \xi, \eta, g)$. Then a necessary and sufficient condition for M to be a \mathcal{PR} -semi-invariant warped product $B \times_f F$ submanifold is that the shape operator of M satisfies*

$$A_{\phi Z} X = -\phi X(\mu)Z, \quad X \in \Gamma(\mathfrak{D} \oplus \langle \xi \rangle), Z \in \Gamma(\mathfrak{D}^\perp), \quad (4.18)$$

for some function μ on M such that $V(\mu) = 0$, $V \in \Gamma(\mathfrak{D}^\perp)$.

Proof. Let $M = B \times_f F$ be a \mathcal{PR} -semi-invariant warped product submanifold of a paracosymplectic manifold \widetilde{M} such that $\xi \in \Gamma(TB)$. Then from Eq. (3.9), we accomplish that $g(A_{\phi Z} W, X) = -(\phi X \ln f)g(W, Z)$ which implies Eq. (4.18). Since f is a function on B , we also have $V(\ln f) = 0$ for all $V \in \Gamma(\mathfrak{D}^\perp)$. Conversely, suppose that M satisfies Eq. (4.18) for some function μ with $V(\mu) = 0$ for all $V \in \Gamma(\mathfrak{D}^\perp)$. Then we have

$$g(h(X, Y), \phi Z) = 0, \quad (4.19)$$

by use of Eqs. (2.5), (2.12) and the fact that \widetilde{M} is a paracosymplectic manifold, we attain that

$$g(\widetilde{\nabla}_X \phi Y, \phi Z) = g(\phi \widetilde{\nabla}_X Y, \phi Z) = -g(\widetilde{\nabla}_X Y, Z) = -g(\nabla_X Y, Z) = 0, \quad (4.20)$$

for any $X, Y \in \Gamma(\mathfrak{D} \oplus \langle \xi \rangle)$, $Z \in \Gamma(\mathfrak{D}^\perp)$. This means that the distribution $(\mathfrak{D} \oplus \langle \xi \rangle)$ is integrable and its leaves are totally geodesic in M . On the other hand, let F be a leaf of \mathfrak{D}^\perp and \mathfrak{h} be the second fundamental form of the

immersion of F into M then for any $Z, W \in \Gamma(\mathfrak{D}^\perp)$, we obtain by using Eqs. (2.9), (2.14) and (4.18) that

$$\mathfrak{h}(Z, W) = g(Z, W)\nabla\mu, \quad (4.21)$$

where $\nabla\mu$ is the gradient of the function μ . Then it follows from (4.21) that the leaves of \mathcal{D}^\perp are totally umbilical in M . Also, for any $V \in \Gamma(\mathfrak{D}^\perp)$, we have $V(\mu) = 0$, which implies that the integral manifold of \mathfrak{D}^\perp is an extrinsic sphere in M , i.e., a totally umbilical submanifold with parallel mean curvature vector. Thus, by [12] we achieve that M is a \mathcal{PR} -semi-invariant submanifold of a paracosymplectic manifold \widetilde{M} . This completes the proof of the theorem. \square

5. Example

In this section, we present an example for a \mathcal{PR} -semi-invariant submanifold of a paracosymplectic manifold in the form $F \times_f B$:

Example 5.1. Let $\widetilde{M} = \mathbb{R}^4 \times \mathbb{R}_+ \subset \mathbb{R}^5$ be a 5-dimensional manifold with the standard Cartesian coordinates (x_1, x_2, y_1, y_2, t) . Define the paracosymplectic pseudo-Riemannian metric structure (ϕ, ξ, η, g) on \widetilde{M} by

$$\phi e_1 = e_3, \phi e_2 = e_4, \phi e_3 = e_1, \phi e_4 = e_2, \phi e_5 = 0, \quad (5.1)$$

$$\xi = e_5, \eta = dt, g = \sum_{i=1}^2 (dx_i)^2 - \sum_{j=1}^2 (dy_j)^2 + \eta \otimes \eta. \quad (5.2)$$

Here, $\{e_1, e_2, e_3, e_4, e_5\}$ is a local orthonormal frame for $\Gamma(T\widetilde{M})$. Let M is an isometrically immersed pseudo-Riemannian submanifold of a paracosymplectic manifold \widetilde{M} given by

$$\Omega(v, \theta, \beta, u) = (v \tan \theta, v \tan \beta, v \sec \theta, v \sec \beta, u),$$

where $\theta \in (0, \pi/2)$, $\beta \in (0, \pi/2)$ and v is non-zero. Then the tangent bundle of M is spanned by the vectors

$$\begin{aligned} X_1 &= \tan(\theta)e_1 + \tan(\beta)e_2 + \sec(\theta)e_3 + \sec(\beta)e_4, \\ X_2 &= v \sec^2(\theta)e_1 + v \sec(\theta) \tan(\theta)e_3, \\ X_3 &= v \sec^2(\beta)e_2 + v \sec(\beta) \tan(\beta)e_4, X_4 = e_5. \end{aligned} \quad (5.3)$$

The space $\phi(TM)$ with respect to the paracosymplectic pseudo-Riemannian metric structure (ϕ, ξ, η, g) of \widetilde{M} becomes

$$\begin{aligned} \phi(X_1) &= \sec(\theta)e_1 + \sec(\beta)e_2 + \tan(\theta)e_3 + \tan(\beta)e_4, \\ \phi(X_2) &= v \sec(\theta) \tan(\theta)e_1 + v \sec^2(\theta)e_3, \\ \phi(X_3) &= v \sec(\beta) \tan(\beta)e_2 + v \sec^2(\beta)e_4, \phi(X_4) = 0. \end{aligned} \quad (5.4)$$

From Eqs.(5.3) and (5.4) we obtain that $\phi(X_4)$ is orthogonal to M , and $\phi(X_1)$, $\phi(X_2)$, $\phi(X_3)$ are tangent to M . So \mathfrak{D}^\perp and \mathfrak{D} can be taken as a subspace $\text{span}\{X_4\}$ and a subspace $\text{span}\{X_1, X_2, X_3\}$ respectively, where $\xi = X_4$ for $\phi(X_4) = 0$ and $\eta(X_4) = 1$. Therefore, M becomes a \mathcal{PR} -semi-invariant submanifold. Further, the induced pseudo-Riemannian metric tensor g of M is given by

$$[g(e_i, e_j)] = \begin{bmatrix} -2 & 0 & 0 & 0 \\ 0 & v^2 \sec^2 \theta & 0 & 0 \\ 0 & 0 & v^2 \sec^2 \beta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

that is,

$$g = du^2 + v^2 \{ \sec^2(\theta)(d\theta)^2 + \sec^2(\beta)(d\beta)^2 - (2/v^2)dv^2 \} = g_F + v^2 g_B.$$

Hence, M is a 4-dimensional \mathcal{PR} -semi-invariant warped product submanifold of \tilde{M} with warping function $f = v^2$.

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